Scaling and rg

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Basics of scaling and overall ideas in RG

Near a critical point only relevant scale for macroscopic properties is the correlation length ξ.

A homogeneous function

$$\psi\left(b^{p_1}x_1, b^{p_2}x_2, \ldots, b^{p_n}x_n\right) = b^{p_f} f\left(x_1, x_2, \ldots, x_n\right)$$

for any real $b^{p_i} \in \mathbb{R}$

Scaling theory: At a critical point free energy density is non-analytic (not divergent)

$$f(t,h) = f_{reg}(t,h) + f_{na}(t,h)$$

remains analytic ↗ ↖ contains the non-analytic term

We chose a notation $t = (T-T_c)/T_c$ and $h = \frac{B-B_c}{B_c}$ with (T_c, B_c) being critical point.

Note that, $t = \left(\frac{T}{J} - \frac{T_c}{J}\right) \Big/ \frac{T_c}{J} = \frac{\frac{1}{J} - \frac{1}{J_c}}{\frac{1}{J_c}} = \frac{J_c - J}{J}$. We shall switch between these two notations time to time.

Widom's scaling hypothesis says

$$f_{na}(t,h) = b\, f_{na}\left(b^{y_1}t, b^{y_2}h\right) \quad \forall\ b, y \in \mathbb{R}.$$

A homogeneous function [any b^y in prefactor can be absorbed by redefining y_1, y_2]

$$= b^n f_{na}\left(b^{ny_1}t, b^{ny_2}h\right)$$

Choosing n such that

$$b^{ny_2}h = 1 \Rightarrow b^n = \frac{1}{h^{1/y_2}}$$

$$\Rightarrow f_{na}(t,h) = \frac{1}{h^{1/y_2}} f_{na}\left(\frac{t}{h^{y_1/y_2}}, 1\right) = \frac{1}{h^{1/y_2}} \cdot \left(\frac{h^{y_1/y_2}}{t}\right)^{1/y_1} \psi\left(\frac{t}{h^{y_1/y_2}}\right)$$

$$\Rightarrow \boxed{f_{na}(t,h) = t^{-1/y_1}\, \psi\left(\frac{t}{h^{y_1/y_2}}\right)}$$

The $\frac{y_2}{y_1} = \Delta$ is called gap exponent.

Remark: In writing f we assume that dimensions are absorbed in a constant prefactor, and that f, t, h all are dimensionless quantities.

Remark: Although the analysis we discuss is for two coupling constants, the analysis is more general.

Relation to standard critical exponents :

summary chart : supported by experimental observations.

Revision of critical exponents : near continuous transition

specific heat	$c \sim \|t\|^{-\alpha}$	for $h=0$
magnetization	$m \sim (-t)^{\beta}$	for $h=0$ and $t<0$
susceptibility	$\chi \sim \|t\|^{-\gamma}$	for $h=0$
magnetization	$\|m\| \sim \|h\|^{1/\delta}$	for $t=0$ and small h

Additional : near critical point, spatial correlation of spins

$$\langle \sigma_r \cdot \sigma_0 \rangle_c \simeq \frac{1}{r^{d-2+\eta}} \cdot e^{-r/\xi}$$

where $\xi \sim |t|^{-\nu}$ is correlation length.

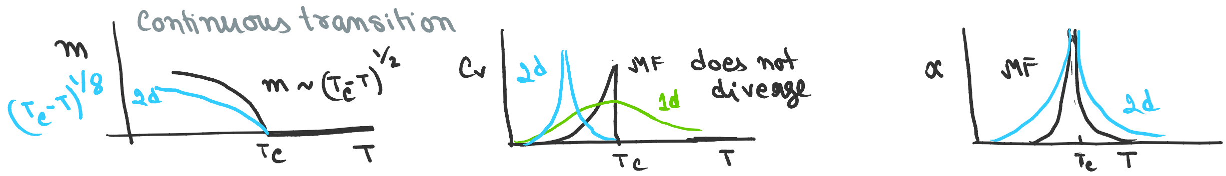
Scaling laws

These exponents are related

$$\alpha + 2\beta + \gamma = 2$$
$$\alpha + \beta\delta + \beta = 2$$
$$\nu(2-\eta) = \gamma$$
$$\alpha + \nu d = 2$$

These come as a result of thermodynamic constraints, such as fluctuation dissipation relation.

An comparative illustration for 2d Ising and mean-field results.



Most of these exponents can be deduced (not quantitatively) from the scaling hypothesis of $f_{na}(t,h)$

Example: the singular part of the specific heat near critical point

$$c(t,h) \sim \frac{\partial f_{na}(t,h)}{\partial t} \sim t^{-\left(2+\frac{1}{y_1}\right)} \Psi_c\left(\frac{h}{t^{\Delta}}\right)$$

$$\Rightarrow \boxed{\alpha = 2 + \frac{1}{y_1}} \qquad \text{with } \Psi_c(0) = \text{constant}$$

Another example :

$$m(t,h) \sim \frac{\partial f_{na}}{\partial h} \sim t^{-\frac{1}{y_1}-\Delta} \Psi_m\left(\frac{h}{t^{\Delta}}\right)$$

$$\Rightarrow m(t,0) \sim t^{-\frac{1}{y_1}-\Delta} \qquad \text{with } \Psi_m(0) = \text{constant}$$

$$\Rightarrow \boxed{\beta = -\frac{1}{y_1} - \Delta}$$

On the other hand, considering m is finite for $t \to 0$ and $h \neq 0$,

$$\Psi_m(x) \sim x^{-\left(\frac{1}{y_1}+\Delta\right)/\Delta} \qquad \text{for } x \to \infty$$

$$\Rightarrow \; m(0,h) \sim h^{-\left(\frac{1}{y_1}+\Delta\right)/\Delta}$$

$$\Rightarrow \; \boxed{\delta = -\frac{\Delta}{\Delta + 1/y_1}} = -\frac{\Delta}{\beta}$$

A combination of all three gives the scaling relation

$$\alpha + \beta\delta + \beta = 2$$

Remark: In a similar way other exponent γ can be determined.

- For the exponent ν we need correlation length.

We expect

$$f_{na} \sim \frac{1}{\xi^d}$$

Then the scaling form of $f_{na}(t,h)$ gives

$$\boxed{\xi \sim t^{\frac{1}{dy_1}} \psi_\xi\left(\frac{h}{t^\Delta}\right)}$$

Immediately gives

$$\boxed{\nu = -\frac{1}{dy_1}}$$

This can be justified by considering the system composed of blocks of size ξ^d such that they are independent. There are $(L/\xi)^d$ blocks each contributing a constant amount in the free energy [an amount equal to free energy at critical point $f(0,0)$, because inside a block is fully correlated]. This is equivalent to saying that leading singularity in free energy density comes from diverging correlation length.

Also confirms the scaling relation

$$\boxed{\alpha + d\nu = 2}$$

Remark: This scaling relation involved dimension d and it is called a hyperscaling relation.

- The exponent η defined in terms of correlation function

$$G(r) \sim \frac{1}{r^{d-2+\eta}} \quad \text{at critical point}$$

is also related by the fluctuation dissipation relation $\chi \sim \int^{\xi} dr\, G(r)$,
[the upper cutoff is used considering slightly away from critical point, and treating distance of size ξ is correlated like exactly at critical point]
which gives

$$\boxed{\gamma = (2-\eta)\nu}$$

Remark: Because we express all these exponents in terms of only two exponents y_1 and y_2, there are only two independent scaling exponents for this theory.

A bit of history: Widom's scaling hypothesis does not by it self give the critical exponents. Later Kadanoff proposed a plausibility argument for his analysis of the Ising model, where he correctly worked on his intuition that divergence of ξ is related to coupling constants of effective Hamiltonian. However, it fell short is finding the flow equations, and did not give the critical exponents. It all came from the RG work of Wilson.